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On the massless scalar field in two space–time dimensions and the Thirring model

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Abstract. In this paper a solution of the renormalised Thirring model constructed only from two scalar massless fields is studied. The representations of the conformal group for the solution as well as those for the scalar fields are obtained. It is proved that the renormalised Thirring equation is covariant with respect to the above representations.

1. Introduction

The free scalar massless field in two space–time dimensions plays an essential role in the construction of the Thirring model. As is well known, such a field does not exist in the conventional sense (the field operators are supposed to act in an indefinite metric space (Wightman 1968, Klaiber 1968)) and it is treated as an auxiliary field. That is why most authors have not paid enough attention to it, and up to recent times some of its properties have remained unknown. We have in mind the existence of a charge that does not annihilate the vacuum state as well as the transformation properties with respect to the two-dimensional conformal group. The existence of such a charge was pointed out in the paper by Nakanishi (1977), but due to the incorrect definition of the dual field the author missed the second charge. The transformation properties with respect to the Lorentz group were studied in the paper by Hadjiivanov and Stoyanov (1977), where the correct representations of the group have been found. Concerning the conformal group, we argue that the investigation of the transformation properties with respect to the latter, performed by Schroer and Swieca (1974), is incomplete. That is why in the present paper a solution of the two-dimensional equations for the free scalar massless field is given. In contrast to Klaiber (1968) and Nakanishi (1977), our formulation is manifestly translationally invariant and the existence of a second charge is taken into account. These solutions are exploited to construct the spinor fields of the Thirring model. The correct treatment of the problems considered gives us the possibility to determine the representation of the conformal group, under which the renormalised Thirring equation is covariant.

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2. The scalar massless field

In this section we define the free quantum scalar massless field in two dimensions. Such a field must satisfy the wave equation

$$\Box\phi(x)=0 \tag{2.1}$$

and the commutation relation

$$[\phi(x),\phi(y)]=\mathrm{i}D(x-y). \tag{2.2}$$

(For the explicit form of $D(x)$ and the other standard commutation functions in two dimensions see appendix 1.)

It is well known that, in two space–time dimensions, a dual scalar field $\tilde{\phi}(x)$ exists, which also satisfies equation (2.1) and (2.2) and is related to $\phi(x)$ by:

$$\partial_\mu\phi(x)+\epsilon_{\mu\nu}\,\partial^\nu\tilde{\phi}(x)=0 \qquad \text{where } \epsilon_{\mu\nu}=-\epsilon_{\nu\mu},\ \epsilon_{01}=1. \tag{2.3}$$

Integrating the differential relation (2.3) we obtain the following explicit form of $\tilde{\phi}(x)$:

$$\tilde{\phi}(x)=\int_{-\infty}^{x^2}\mathrm{d}z^1\,\partial_0\phi(x^0,z^1)+R(x^0), \tag{2.4}$$

where $R(x^0)$ is a regularisation counter-term for the integral on the right-hand side of equation (2.4), which in general is divergent. One can prove that $R(x^0)$ must satisfy the following reltaions:

$$\partial_0R(x^0)=\partial_1\phi(x^0,x^1)|_{x^1\to-\infty} \qquad [R(x^0),\phi(y)]=d=\text{constant}. \tag{2.5}$$

Then equations (2.4) and (2.5) imply that the following commutation relation between $\tilde{\phi}(x)$ and $\phi(y)$ holds:

$$[\tilde{\phi}(x),\phi(y)]=\mathrm{i}\tilde{D}(x-y)-\tfrac{1}{2}\mathrm{i}+d. \tag{2.6a}$$

So we see that the regularisation procedures leads to an arbitrariness in the commutation relation (2.6a). In what follows we fix the latter by choosing $d=\frac{1}{2}\mathrm{i}$, i.e.

$$[\tilde{\phi}(x)_1\phi(y)]=\mathrm{i}\tilde{D}(x-y). \tag{2.6}$$

In order to separate the creation and annihilation parts from the fields $\phi(x)$ and $\tilde{\phi}(x)$ one must either use the formula

$$\phi^\pm(x)=-\mathrm{i}\int\mathrm{d}z^1D^\pm(x-z)\,\overleftrightarrow{\partial}{}_0^z\phi(z) \tag{2.7}$$

(and the analogous formula for the field $\tilde{\phi}(x)$), or define the momentum space representation. As is well known, the Fourier integral in that case is divergent, due to the infrared singularity of the Lorentz invariant measure. The regularisation procedure can be determined by the condition that the two methods for separation of the creation and annihilation parts are compatible.

The mathematical peculiarities of the two-dimensional scalar massless fields require their more precise definition as operator-valued generalised functions. Namely, we consider those solutions of equations (2.1)–(2.6) which satisfy the following subsidiary conditions.

(a) The fields $\phi(x)$ and $\tilde{\phi}(x)$ as well as their creation and annihilation parts are integrable in the space $S(\mathbb{R}_2)$, which consists of complex infinitely differentiable and rapidly decreasing functions of two variables.

(*b*) The integrals

$$A^{\pm}(0) = \mathrm{i}(2/\pi)^{1/2} \int \partial_0 \phi^{\pm}(x)\, \mathrm{d}x^1 \tag{2.8}$$

$$B^{\pm}(0) = -\mathrm{i}(2/\pi)^{1/2} \int \partial_0 \tilde{\phi}^{\pm}(x)\, \mathrm{d}x^1 \tag{2.9}$$

are convergent. This condition is relevant to the regularisation procedure, as well as to the existence of non-zero charges. As is well known the quantities

$$Q = \mathrm{i}(\pi/2)^{1/2}(A^-(0) - A^+(0)) \tag{2.10}$$

$$\tilde{Q} = -\mathrm{i}(\pi/2)^{1/2}(B^-(0) - B^+(0)) \tag{2.11}$$

are conserved. One can readily prove that they do not commute with the fields $\phi(x)$ and $\tilde{\phi}(x)$ due to the equal-time canonical commutators

$$[\partial_0\phi(x^0, x^1), \phi(x^0, y^1)] = -\mathrm{i}\, \delta(x^1 - y^1). \tag{2.12}$$

That is why the quantities Q and $\tilde{Q}$ should not vanish. One can esaily see, integrating the above formula over x^1, that the following commutation relations hold:

$$[Q, \phi(x)] = [\tilde{Q}, \tilde{\phi}(x)] = -\mathrm{i}, \tag{2.13}$$

while the remaining two commutators are zero due to the fact that $\partial_0 \tilde{D}(x)|_{x^0=0} = 0$.

Now we give the definition of the fields $\phi^{\pm}(x)$ and $\tilde{\phi}^{\pm}(x)$ as operator-valued generalised functions in $S(\mathbb{R}_2)$. Namely, we fix the regularisation of the infrared divergence by defining the Fourier transforms of the fields in the following way. Suppose $\Phi(x)$ belongs to the $S(\mathbb{R}_2)$ space and $F(p)$ is its Fourier transform

$$F(p^0, p^1) = \frac{1}{2\pi} \int \Phi(x)\, \mathrm{e}^{-\mathrm{i}px}\, \mathrm{d}^2x, \tag{2.14}$$

while

$$f^{\pm}(p^1) = F(\pm|p'|, \pm p^1). \tag{2.15}$$

Then we define

$$\phi^{\pm}(\Phi) \equiv \int \phi^{\pm}(x)\Phi(x)\, \mathrm{d}^2x = \left(\frac{\pi}{2}\right)^{1/2} \int \frac{\mathrm{d}p^1}{|p^1|}(A^{\pm}(p^1)f^{\pm}(p^1) - A^{\pm}(0)f^{\pm}(0)\theta(\kappa - |p^1|)), \tag{2.16}$$

$$\tilde{\phi}^{\pm}(\Phi) \equiv \int \tilde{\phi}^{\pm}(x)\Phi(x)\, \mathrm{d}^2x = -\left(\frac{\pi}{2}\right)^{1/2} \int \frac{\mathrm{d}p^1}{|p^1|}(B^{\pm}(p^1)f^{\pm}(p^1) - B^{\pm}(0)f^{\pm}(0)\theta(\kappa - |p^1|)). \tag{2.17}$$

It is readily seen that out regularisation is translationally invariant, contrary to that of Klaiber (1968). The latter has the following form in our notation:

$$\phi^{\pm}_{\mathrm{K}}(\phi) \equiv \int \phi^{\pm}_{\mathrm{K}}(x)\Phi(x)\, \mathrm{d}^2x = \left(\frac{\pi}{2}\right)^{1/2} \int \frac{\mathrm{d}p^1}{|p^1|} A^{\pm}(p^1)(f^{\pm}(p^1) - f^{\pm}(0)\theta(\kappa - |p^1|)), \tag{2.18}$$

and an analogous formula for the fields $\tilde{\phi}^{\pm}_{\mathrm{K}}(x)$. (Here the subscript K stands for

Klaiber's fields.) Evidently formulae (2.16) and (2.18) differ by the constant operator

$$\phi^{\pm}(\Phi)-\phi^{\pm}_{\mathrm{K}}(\Phi)=\left(\frac{\pi}{2}\right)^{1/2}f^{\pm}(0)\int_{-\kappa}^{\kappa}\frac{A^{\pm}(p^1)-A^{\pm}(0)}{|p^1|}\,\mathrm{d}p^1=f^{\pm}(0)G^{\pm}. \quad (2.19)$$

Analogously

$$\tilde{\phi}^{\pm}(\Phi)-\tilde{\phi}^{\pm}_{\mathrm{K}}(\Phi)=-\left(\frac{\pi}{2}\right)^{1/2}f^{\pm}(0)\int_{-\kappa}^{\kappa}\frac{B^{\pm}(p^1)-B^{\pm}(0)}{|p^1|}\,\mathrm{d}p^1=f^{\pm}(0)\tilde{G}^{\pm}. \quad (2.20)$$

The operators (2.19) and (2.20) are obviously translationally non-invariant. It is evident that if the test functions $\Phi(x)$ satisfy the condition

$$\int \Phi(x)\,\mathrm{d}^2x=2\pi f^{\pm}(0)=2\pi f(0)=0 \quad (2.21)$$

then our regularisation and that of Klaiber are identical. (We denote the subspace of functions $f(p^1)\in S(\mathbb{R}_2)$ for which $f(0)=0$ by $S_0(C_+)$.) It is readily seen that on $S_0(C_+)$ the metric is positively definite, since the regularising term that makes it indefinite drops out on $S_0(C_+)$. Besides that, if the fields are defined as operator-valued functions on $S_0(C_+)$ then the charges Q and $\tilde{Q}$ should commute with the field which is not contradictory to the fact that they are non-zero.

The commutators $[A^+(p^1), A^-(q^1)]$ and $[B^+(p^1), B^-(q^1)]$ differ from the canonical ones by a counter-term with support at the point $p^1=q^1=0$ (see appendix 2).

One can easily check that the condition that $\tilde{Q}$ is non-zero implies that $A^{\pm}(p^1)$ has a jump at the point $p^1=0$, i.e.

$$A^{\pm}(p^1)=a^{\pm}(p^1)+\epsilon(p^1)f^{\pm}(p^1). \quad (2.22)$$

where $a^{\pm}$ and $b^{\pm}$ are new creation and annihilation operators, while $\epsilon(p^1)$ is the usual sign function defined at the point $p^1=0$ by $\epsilon(0)=0$, so that the integrals, where $\epsilon(p^1)$ occur, are defined in the sense of principal value $(\epsilon(p^1)/|p^1|=\mathscr{P}(1/p^1))$. The differential relation between $\phi(x)$ and $\tilde{\phi}(x)$ leads to the equality

$$B^{\pm}(p^1)=b^{\pm}(p^1)+\epsilon(p^1)a^{\pm}(p^1). \quad (2.23)$$

Now the operators $A^{\pm}(p^1)$ and $B^{\pm}(p^1)$ are completely determined if we determine their behaviour in the neighbourhood of the point $p^1=0$. We stress that $A^{\pm}(0)=a^{\pm}(0)$ and $B^{\pm}(0)=b^{\pm}(0)$ must be the limits of the operators $a^{\pm}(p^1)$ and $b^{\pm}(p^1)$ when $p^1\to 0$. So that we can write

$$\left.\begin{aligned}a^{\pm}(p^1)&\sim a^{\pm}(0)+|p^1|^{\alpha}c^{\pm}(p^1)\\ b^{\pm}(p^1)&\sim b^{\pm}(0)+|p^1|^{\tilde{\alpha}}\tilde{c}^{\pm}(p^1)\end{aligned}\right\}\qquad p^1\sim 0 \quad (2.24)$$

where $\mathrm{Re}\,\overset{(\sim)}{\alpha}>0$ and

$$\lim_{p^1\to 0}|p^1|^{\alpha}c^{\pm}(p^1)=\lim_{p^1\to 0}|p^1|^{\tilde{\alpha}}\tilde{c}^{\pm}(p^1)=0.$$

In what follows, we shall operate with normal ordered products of these fields as well as their normal ordered exponentials. Since the problem of their definition is discussed in detail in Wightman (1968) and Klaiber (1968) we shall only note here that normal ordered products and exponentials of our fields can be expressed by those of Klaiber in view of equations (2.19) and (2.20).

3. The massless Thirring model

In this section we consider the renormalised massless Thirring model. Recently Mandelstam (1975) and Pogrebkov and Sushko (1975, 1976) have shown how a spinor field can be constructed from exponentials of scalar fields. So we argue that the solution of the renormalised Thirring model can be built from exponentials of the fields $\phi^{\pm}(x)$ and $\tilde{\phi}^{\pm}(x)$ only. We do not discuss here the problem of the existence of exponentials of our scalar fields, since, as was pointed out in the previous section, there is no difference in principle between solving the problem in our case and that of Klaiber.

Now we formulate the following statement. Consider the quantities

$$(\psi(x))_k = (e^{i\beta\gamma^5\tilde{\phi}^-(x)}\, e^{-i\alpha\phi^-(x)}\, e^{-i\alpha\phi^+(x)}\, e^{i\beta\gamma^5\tilde{\phi}^+(x)} u)_k$$
$$= e^{i\beta(-1)^k\tilde{\phi}^-(x)}\, e^{-i\alpha\phi^-(x)}\, e^{-i\alpha\phi^+(x)}\, e^{i\beta\gamma^5\tilde{\phi}^+(x)} u_k, \tag{3.1}$$

where

$$|u_1|^2 = |u_2|^2 = \frac{1}{2\pi}(\kappa\, e^{-\Gamma'(1)})(\alpha^2+\beta^2)/4\pi, \qquad u_k, \quad k=1,2$$

are complex numbers and

$$\alpha = \left(\pi\frac{1-h}{1+h}\right)^{1/2}, \qquad \beta = \left(\pi\frac{1+h}{1-h}\right)^{1/2}, \qquad h = \frac{g}{2\pi}. \tag{3.2}$$

Suppose that $\phi^{\pm}(x)$ and $\tilde{\phi}^{\pm}(x)$ satisfy equations (2.1)–(2.3) and (2.6), then the quantity (3.1) satisfies the renormalised equation of the Thirring model:

$$i\gamma^\mu\, \partial_\mu\psi(x) = -g{:}\, \mathcal{J}_\mu(x)\gamma^\mu\psi(x){:} \tag{3.3}$$

where the Dirac matrices γ^μ are chosen to be $\gamma^0=\sigma_1$, $\gamma^1 = i\sigma_2$, $\gamma^5\equiv\gamma^0\gamma^1 = -\sigma_3$ and σ_i, $i=1,2,3$, are the Pauli matrices, while the renormalized current $\mathcal{J}_\mu(x)$ is defined by (see Johnson (1961)

$$\mathcal{J}_\mu(x) = \tfrac{1}{2}(j_\mu(x)+\tilde{j}_\mu(x)), \tag{3.4}$$

$j_\mu(x)$ and $\tilde{j}_\mu(x)$ being

$$j_\mu(x) = \lim_{\substack{\epsilon^0=0\\ \epsilon^1\to 0}} j_\mu(x,\epsilon) = \lim_{\substack{\epsilon^0=0\\ \epsilon^1\to 0}} (-\epsilon^2)^{(\alpha^2+\beta^2)/4\pi-\frac{1}{2}}[\bar{\psi}(x+\epsilon)\gamma_\mu\psi(x) - \psi(x)(\bar{\psi}(x-\epsilon)\gamma_\mu)], \tag{3.5}$$

$$\tilde{j}_\mu(x) = \lim_{\substack{\epsilon^0=0\\ \epsilon^1\to 0}} \tilde{j}_\mu(x,\epsilon) = \lim_{\substack{\epsilon^0=0\\ \epsilon^1\to 0}} j_\mu(x,\tilde{\epsilon}), \tag{3.6}$$

where

$$\tilde{\epsilon}^2 = -\epsilon^2 \qquad \tilde{\epsilon}.\,\epsilon = 0. \tag{3.7}$$

The proof goes as follows. We first calculate the current $\mathcal{J}_\mu(x)$. For this purpose we substitute (3.1) into (3.5) and (3.6), and make the necessary re-ordering. Taking the limits we obtain

$$j_\mu(x) = \frac{i}{2\pi}\sum_{k=1}^{2}(-1)^{\mu(k+)+1+\frac{1}{2}(-1)^k}\,\delta_1 F_k(x). \tag{3.8}$$

$$\tilde{j}_\mu(x)=\frac{\mathrm{i}}{2\pi}\sum_{k=1}^{2}(-1)^{\mu(k+1)+\frac{1}{2}}\,\partial_0F_k(x),\tag{3.9}$$

where $F_k(x)=\alpha\phi(x)+(-1)^k\beta\tilde{\phi}(x)$, and we have used the fact that $\alpha\beta=\pi$.

Inserting (3.8) and (3.9) into (3.4) we obtain

$$\mathcal{J}_\mu(x)=\frac{1}{2\pi}(\alpha+\beta)\,\partial_\mu\phi.\tag{3.10}$$

On the other hand using (2.3) and the equality $\gamma_\mu\gamma^5=\epsilon_{\mu\nu}\gamma^\nu$ one can easily check the identity

$$\mathrm{i}\gamma_\mu\partial_\mu\psi(x)=(\alpha-\beta)\gamma^\mu{:}\partial_\mu\phi(x)\psi(x){:}.\tag{3.11}$$

Therefore substituting (3.10) and (3.11) into (3.3) we find an algebraic equation for the coefficients α and β, which, as we can easily see, is satisfied by α and β from equation (3.2).

Therefore $\psi(x)$ defined by (3.1) and (3.2) is a solution of the renormalized Thirring model. Comparing it with the solution of Klaiber, we see that they coincide on $S_0(C_+)$.

4. Transformation laws for the fields $\phi^{\pm}(x)$, $\tilde{\phi}^{\pm}(x)$ and $\psi(x)$

In this section we obtain representations of the two-dimensional conformal group, which act in the space of the field operators $\phi^{\pm}(x)$, $\tilde{\phi}^{\pm}(x)$ and therefore of $\psi(x)$. One could try to proceed directly in order to construct transformations, which leave all commutators of the scalar fields (equal frequency commutators included) invariant. Since such an approach is more difficult, we use another; one which has the advantage of showing that the structure of the scalar field obtained in § 2 leads to these transformations. We begin with the Lorentz transformations. In Schroer and Swieca (1974) it was proved that under Lorentz transformations the fields $\phi^{\pm}(x)$ and $\tilde{\phi}^{\pm}(x)$ transform inhomogeneously:

$$\phi^{\pm}(x)\xrightarrow{\Lambda_\chi}\phi^{\pm}(\Lambda_\chi x)-\frac{\chi}{2(2\pi)^{1/2}}b^{\pm}(0),\tag{4.1}$$

$$\Lambda_\chi=\begin{pmatrix}\cosh\chi & \sinh\chi\\ \sinh\chi & \cosh\chi\end{pmatrix}.$$

$$\tilde{\phi}^{\pm}(x)\xrightarrow{\Lambda_\chi}\tilde{\phi}^{\pm}(\Lambda_\chi x)+\frac{\chi}{2(2\pi)^{1/2}}a^{\pm}(0),\tag{4.2}$$

Now since χ (the parameter of the Lorentz transformation) and $b^{\pm}(0)$ are pseudoscalars, while $a^{\pm}(0)$ is a scalar, we see that relations (4.1) and (4.2) have the proper space parity. Another feature is that they do not mix creation and annihilation parts. Applying these transformations to the quantity $\psi(x)$ defined by (3.1), we obtain

$$\psi(x)\xrightarrow{\Lambda_\chi}{:}\mathrm{e}^{\mathrm{i}\chi(\alpha S+\beta\gamma_5L)}\psi(\Lambda_\chi x){:}\tag{4.3}$$

where the operators L and S are defined as

$$L=\frac{1}{2(2\pi)^{1/2}}(a^+(0)+a^-(0)),\qquad S=\frac{1}{2(2\pi)^{1/2}}(b^+(0)+b^-(0)).\tag{4.4}$$

We see that equation (4.3) is not a transformation law for a spinor.

One can easily check that the quantity $\bar{\psi}\psi$ is not Lorentz invariant, but the quantity $\bar{\psi}\gamma^{\mu}\psi$ is a Lorentz vector. Thus equation (4.3) naturally distinguishes between the scalar and the vector forms of the Thirring model.

Now we consider the dilations. Since the commutation function $D^{\pm}(x)$ is not invariant, i.e. it transforms under dilations as

$$D^{\pm}(x) \xrightarrow{D_\lambda} D^{\pm}(\lambda x) = D^{\pm}(x) - \frac{1}{2\pi} \ln \lambda, \tag{4.5}$$

we see that the proper definitions of the transformation laws of $\varphi^{\pm}(x)$ and $\tilde{\varphi}^{\pm}(x)$ are inhomogenous, the inhomogenous terms being linear combinations of the constant operators $a^{\pm}(0)$ and $b^{\pm}(0)$, as they are in equation (4.1) and (4.2). One can readily observe that the coefficients of this combination are not uniquely determined by the condition that the commutators of $\varphi^{\pm}(x)$ and $\tilde{\varphi}^{\pm}(x)$ are invariant under the dilatation group, and therefore an arbitrariness exists. We fix this arbitrariness by imposing the conditions that the transformed fields be scalar and pseudoscalar again, and that the following equations:

$$\varphi^{+}(x)|0\rangle = \tilde{\varphi}^{+}(x)|0\rangle = 0, \tag{4.6}$$

be invariant under the group. Both these conditions have a group meaning, the first because space parity is included in the full conformal group, while the group character of the second can be seen after considering the special conformal transformations (at least infinitesimally).

Thus we can write finally

$$\varphi^{\pm}(x) \xrightarrow{D_\lambda} \varphi^{\pm}(\lambda x) + \frac{a^{\pm}(0)}{2(2\pi)^{1/2}} \ln \lambda, \tag{4.7}$$

$$\tilde{\varphi}^{\pm}(x) \xrightarrow{D_\lambda} \tilde{\varphi}^{\pm}(\lambda x) - \frac{b^{\pm}(0)}{2(2\pi)^{1/2}} \ln \lambda. \tag{4.8}$$

Inserting equations (4.7) and (4.8) in (3.1) we obtain

$$\psi(x) \xrightarrow{D_\lambda} :\exp[\mathrm{i} \ln \lambda\, (\alpha L + \beta \gamma^5 S)]\psi(\lambda x):. \tag{4.9}$$

The transformation law (4.9) is not the standard one and therefore it is obvious that we cannot assign any conformal dimension to the field $\psi(x)$, although such a dimension appears in the two-point function. This is a very important feature because even if we use the standard formulation of the Thirring field (i.e. define $\psi(x)$ in the Hilbert space) the transformation law (4.9) remains valid.

We now consider the special conformal transformations. It is easier to begin with the infinitesimal transformations, so first we give some definitions. We define the representation acting in the space of field operators as

$$U_g^{-1}\phi(x)U_g = T_g\phi(x), \tag{4.10}$$

where, for the Lorentz group, T_g is defined by the right-hand sides of equation (4.1) or (3.2), while, for the dilations, by the right-hand sides of equations (4.7) or (4.8) respectively. We define the generators by

$$I_g = -\mathrm{i} \frac{\partial U_g}{\partial \alpha_g}\bigg|_{\alpha=0}; \tag{4.11}$$

particularly we have

$$M_{\mu\nu}=\mathrm{i}\epsilon_{\mu\nu}\frac{\partial U_\chi}{\partial\chi}\bigg|_{\chi=0} \tag{4.12}$$

Now we can write down the commutators of the generators of the representations (4.1), (4.2), (4.7) and (4.8) with the fields $\phi^{\pm}(x)$ and $\tilde{\phi}^{\pm}(x)$, respectively

$$[M_{\mu\nu},\phi^{\pm}(x)]=-\mathrm{i}(x_\mu\,\partial_\nu-x_\nu\,\partial_\mu)\phi^{\pm}(x)+\frac{\mathrm{i}}{2(2\pi)^{1/2}}\epsilon_{\mu\nu}b^{\pm}(0), \tag{4.13}$$

$$[M_{\mu\nu},\tilde{\phi}^{\pm}(x)]=-\mathrm{i}(x_\mu\,\partial_\nu-x_\nu\,\partial_\mu)\tilde{\phi}^{\pm}(x)-\frac{\mathrm{i}}{2(2\pi)^{1/2}}\epsilon_{\mu\nu}a^{\pm}(0), \tag{4.14}$$

$$[D,\phi^{\pm}(x)]=\mathrm{i}x^\mu\,\partial_\mu\phi^{\pm}(x)+\frac{\mathrm{i}}{2(2\pi)^{1/2}}a^{\pm}(0), \tag{4.15}$$

$$[D,\tilde{\phi}^{\pm}(x)]=\mathrm{i}x^\mu\,\partial_\mu\tilde{\phi}^{\pm}(x)-\frac{\mathrm{i}}{2(2\pi)^{1/2}}b^{\pm}(0). \tag{4.16}$$

Having in mind formulae (4.13)–(4.16) we can realise the generators K_μ of the special conformal transformations by the following formulae:

$$[K_\mu,\phi^{\pm}(x)]=-\mathrm{i}(2x_\mu x_\nu-g_{\mu\nu}x^2)\,\partial^\nu\phi^{\pm}(x)-\frac{\mathrm{i}x_\mu}{(2\pi)^{1/2}}a^{\pm}(0)+\frac{\mathrm{i}\epsilon_{\mu\nu}x^\nu}{(2\pi)^{1/2}}b^{\pm}(0), \tag{4.17}$$

$$[K_\mu,\tilde{\phi}^{\pm}(x)]=-\mathrm{i}(2x_\mu x_\nu-g_{\mu\nu}x^2)\,\partial^\nu\tilde{\phi}^{\pm}(x)+\frac{\mathrm{i}x_\nu}{(2\pi)^{1/2}}b^{\pm}(0)-\frac{\mathrm{i}\epsilon_{\mu\nu}x^\nu}{(2\pi)^{1/2}}a^{\pm}(0). \tag{4.18}$$

In order to prove formulae (4.17) and (4.18) one must check that the commutators of K_μ with $M_{\mu\nu}$, D and expecially with $P_\mu(P_\mu=\mathrm{i}\,\partial_\mu)$ are the correct ones for the conformal algebra in two dimensions. Since this is trivial we omit it here.

Now we shall reconstruct the global transformations. For the purpose, we note that in the two-dimensional space–time the conformal group is decomposed into a direct product of two groups SL(2, $\mathbb{R}$). Indeed if we denote

$$x_+=x^0+x^1,\qquad x_-=x^0-x^1, \tag{4.19}$$

then x_+ and x_- form invariant subspaces, i.e. they transform as follows under special conformal transformations:

$$x_+\to\frac{x_+}{\rho_+(\delta,x_+)},\qquad x_-\to\frac{x_-}{\rho_-(\delta,x_-)}, \tag{4.20}$$

where

$$\rho_\pm(\delta,x_\pm)=1+(\delta^0\mp\delta^1)x_\pm, \tag{4.21}$$

and δ^μ are the parameters of the special conformal transformations. One can see that the quantities $\ln\rho=\ln(\rho_+\rho_-)$ and $\ln\sigma=\ln(\rho_+/\rho_-)$ have the space parity of a scalar and a pseudoscalar, respectively, and that they satisfy the following equations:

$$\frac{\partial\ln\rho}{\partial\delta^\mu}\bigg|_{\delta^\mu=0}=2x_\mu\qquad\frac{\partial\ln\sigma}{\partial\delta^\mu}\bigg|_{\delta^\mu=0}=-2\epsilon_{\mu\nu}x^\nu. \tag{4.22}$$

So if we write

$$\phi^{\pm}(x) \xrightarrow{K_\delta} \phi^{\pm}\left(\frac{x_\mu + \delta_\mu x^2}{\rho(\delta, x)}\right) - \frac{a^{\pm}(0)}{2(2\pi)^{1/2}} \ln|\rho(\delta, x)| + \frac{b^{\pm}(0)}{2(2\pi)^{1/2}} \ln|\sigma(\delta, x)|, \tag{4.23}$$

$$\tilde{\phi}^{\pm}(x) \xrightarrow{K_\delta} \tilde{\phi}^{\pm}\left(\frac{x_\mu + \delta_\mu x^2}{\rho(\delta, x)}\right) + \frac{b^{\pm}(0)}{2(\pi)^{1/2}} \ln|\rho(\delta, x)| - \frac{a^{\pm}(0)}{2(2\pi)^{1/2}} \ln|\sigma(\delta, x)|, \tag{4.24}$$

one can easily prove that the infinitesimal transformations following from (4.23) and (4.24) coincide with those implied by equations (4.17) and (4.18), respectively. It is also easy to check that the transformations (4.23) and (4.24) have the necessary group property, so we omit the proof here.

Now we can define the special conformal transformations of the Thirring field. They read

$$\psi(x) \xrightarrow{K_\delta} :|\rho(\delta, x)|^{\mathrm{i}(\alpha L + \beta\gamma^5 S)} |\sigma(\delta, x)|^{\mathrm{i}(\alpha S + \beta\gamma^5 L)} \psi\left(\frac{x_\mu + \delta_\mu x^2}{\rho(\delta, x)}\right):. \tag{4.25}$$

The commutator of the corresponding generators with $\psi(x)$ is then

$$\begin{aligned}[K_\mu, \psi(x)] = -\mathrm{i}(2x_\mu x_\nu - g_{\mu\nu}x^2)\partial^\nu\psi(x) - 2x_\mu :(\alpha L + \beta\gamma^5 S)\psi(x): \\ + 2\epsilon_{\mu\nu}x^\nu :(\alpha S + \beta\gamma^5 L)\psi(x):. \end{aligned} \tag{4.26}$$

At this point we must not that representations, similar to those given by formula (4.23), have already been discussed by Schroer and Swieca (1974) but they missed they missed the term $\ln|\sigma(\delta, x)|$ (in our notation). We insist that the latter is very important, as we shall see in the following section.

5. The covariance of the renormalised Thirring equation

In this section we prove that the renormalised Thirring equation is covariant under the representations of the conformal group obtained in the previous section.

We first note that all commutators of the fields $\phi^{\pm}(x)$ and $\tilde{\phi}^{\pm}(x)$ are invariant, while equations (2.1) and (2.3) are covariant under the action of the Lorentz transformations (equations (4.1) and (4.2)), the dilatations (equations (4.7) and (4.8)) and the infinitesimal special conformal transformations (equations (4.17) and (4.18)). This is easily verified by direct computation. We shall show that this property takes place even for representations of a larger group, namely the universal covering group of the conformal group.

For this purpose, we consider the variables x_+ and x_- defined by equation (4.19) as complex, and in what follows we denote them by z_+ and z_-, respectively. As we mentioned in the previous section, the conformal group, with respect to the variables $z_\pm$, is decomposed into two SL(2, $\mathbb{R}$) groups, which act on $z_\pm$ in the following way:

$$z_\pm \to z'_\pm = \frac{\alpha z_\pm + \delta}{\beta z_\pm + \gamma}, \tag{5.1}$$

where $\alpha\gamma - \beta\delta = 1$, α, β, γ and δ being real parameters. (We must note that for these variables the proper Lorentz group action and the dilatations coincide, the parameters being $\gamma = 1$, $\beta = \delta = 0$, $\alpha = \mathrm{e}^{\pm\chi}(=\lambda)$, while for translations we have parameters $\beta = 0$, $\alpha = \gamma = 1$.) It is well known that the representation of SL(2, $\mathbb{R}$) given by formula (5.1) conserves the upper (lower) complex half-plane. So that if we introduce the functions

$$\mathbb{D}(z) = -\ln(-\mu^2 z_+ z_-)/4\pi \tag{5.2}$$

and

$$\tilde{\mathbb{D}}(z) = \frac{1}{4\pi} \ln\left(\frac{z_-}{z_+}\right), \tag{5.3}$$

then it is meaningful to define

$$D^{\pm}(x) = \pm \lim_{\mathrm{Im}\, z \to \mp \mathrm{i}0} \mathbb{D}(z), \tag{5.3a}$$

$$\tilde{D}^{\pm}(x) = \pm \lim_{\mathrm{Im}\, z \to \mp \mathrm{i}0} \tilde{\mathbb{D}}(z). \tag{5.3b}$$

Since $\mathbb{D}(z)$ and $\tilde{\mathbb{D}}(z)$ under special conformal transformations ($\alpha = \gamma = 1$, $\delta = 0$, $\beta = \delta_{\pm}$) transform as

$$\mathbb{D}(z) \to \mathbb{D}(z) + \frac{1}{4\pi} \ln \rho(\delta, z), \tag{5.4}$$

$$\tilde{\mathbb{D}}(z) \to \tilde{\mathbb{D}}(z) + \frac{1}{4\pi} \ln \sigma(\delta, z). \tag{5.5}$$

we see that we can keep the commutators of $\phi^{\pm}(x)$ and $\tilde{\phi}^{\pm}(x)$ invariant with respect to the latter, if instead of equations (4.23) and (4.24) we take the following transformation laws:

$$\phi^{\pm}(x) \xrightarrow{K_\delta} \phi^{\pm}\left(\frac{X_\mu + \delta_\mu x^2}{\rho(\delta, x)}\right) - \frac{a^{\pm}(0)}{2(2\pi)^{1/2}} \ln \rho(\delta, x) + \frac{b^{\pm}(0)}{2(2\pi)^{1/2}} \ln \sigma(\delta, x), \tag{5.6}$$

$$\tilde{\phi}^{\pm}(x) \xrightarrow{K_\delta} \tilde{\phi}^{\pm}\left(\frac{x_\mu + \delta_\mu x^2}{\rho(\delta, x)}\right) + \frac{b^{\pm}(0)}{2(2\pi)^{1/2}} \ln \rho(\delta, x) - \frac{a^{\pm}(0)}{2(2\pi)^{1/2}} \ln \sigma(\delta, x), \tag{5.7}$$

with $x^0 \to x^0 - \mathrm{i}0$ for the annihilation operators and $x^0 \to x^0 + \mathrm{i}0$ for the creation operators.

Now one can easily prove that the field equations (2.1) and (2.3) are covariant with respect to the transformation laws (5.6) and (5.7) since $\ln \rho$ and $\ln \sigma$ satisfy

$$\Box \ln \rho = \Box \ln \sigma = 0, \tag{5.8}$$

$$\partial_\mu \ln \rho + \epsilon_{\mu\nu} \partial^\nu \ln \sigma = 0. \tag{5.9}$$

Therefore if $\phi^{\pm}(x)$ and $\tilde{\phi}^{\pm}(x)$ are transformed according to the representations (5.6) and (5.7) of the universal covering group of the conformal group, then the transformed fields satisfy equations (2.1)–(2.3) and (2.6). We now write, for completeness, the transformation law for the Thirring field

$$\psi(x) \xrightarrow{K_\delta} :\rho(\delta, x)^{\mathrm{i}(\alpha L + \beta\gamma^5 S)} \sigma(\delta, x)^{\mathrm{i}(\alpha S + \beta\gamma^5 L)} \psi\left(\frac{x_\mu + \delta_\mu x^2}{\rho(\delta, x)}\right):. \tag{5.10}$$

We note that in the case of equation (5.10) the normal ordering means not only that creation operators are kept to the left and annihilation operators to the right, but also that these operators have the proper sign of i0 from equations (5.6) and (5.7).

If we now recall the statement proved in § 3, we see that the quantity (3.1) constructed from the transformed fields $(\phi^{\pm}(x))'$ and $(\tilde{\phi}^{\pm}(x))'$ (i.e. that defined by equation (5.10)) is again a solution of the Thirring equation (3.3), since it is shown that the transformed fields $(\phi^{\pm}(x))'$ and $(\tilde{\phi}^{\pm}(x))'$ satisfy the same equation as the original ones. Therefore, the Thirring equation is covariant with respect to representations of the universal covering group of the conformal group.

So we see that, despite their quite peculiar form, the representations obtained in this paper proved to be sensible indeed.

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Appendix 1. Summary of the definitions of the commutation functions $D(x)$, $\tilde{D}(x)$, $D^{\pm}(x)$ and $\tilde{D}^{\pm}(x)$

$$D(x)=\frac{1}{2\pi\mathrm{i}}\int \mathrm{d}^2p\,\epsilon(p^0)\delta(p^2)\,\mathrm{e}^{-\mathrm{i}px}=-\frac{1}{2}(\theta(x^1+x^0)-\theta(x^1-x^0))=-\frac{1}{2}\epsilon(x^0)\theta(x^2)$$

$$\mathrm{i}D(x)=D^+(x)+D^-(x)$$

$$D^{\pm}(x)=\pm\frac{1}{4\pi}\int\frac{\mathrm{d}p^1}{|p^1|}[\mathrm{e}^{\mp\mathrm{i}px}-\theta(\kappa-|p^1|)]_{p^0=|p^1|}$$

$$=\mp\frac{1}{4\pi}\ln(-\mu^2x^2\pm\mathrm{i}0x^0)$$

$$=\mp\frac{1}{4\pi}\ln\mu^2|x^2|-\frac{\mathrm{i}}{4}\epsilon(x^0)\theta(x^2),\qquad \mu=\kappa\,\mathrm{e}^{-\Gamma'(1)}$$

$$\tilde{D}(x)=-\frac{1}{2\pi\mathrm{i}}\int \mathrm{d}^2p\,\epsilon(p^1)\delta(p^2)\,\mathrm{e}^{-\mathrm{i}px}=-\frac{1}{2}(\theta(x^0+x^1)-\theta(x^0-x^1))=-\frac{1}{2}\epsilon(x^1)\theta(-x^2)$$

$$\mathrm{i}\tilde{D}(x)=\tilde{D}^+(x)+\tilde{D}^-(x)$$

$$\tilde{D}^{\pm}(x)=\mp\mathcal{P}\int\frac{\mathrm{d}p^1}{p^1}\,\mathrm{e}^{\mp\mathrm{i}px}$$

$$=\pm\frac{1}{4\pi}\ln\frac{x^0-x^1-\mathrm{i}0}{x^0+x^1-\mathrm{i}0}$$

$$=\pm\frac{1}{4\pi}\ln\left|\frac{x^0-x^1}{x^0+x^1}\right|-\frac{\mathrm{i}}{4}\epsilon(x^1)\theta(-x^2)$$

Appendix 2. Summary of some useful commutators

$$[A^+(p), A^-(q)] = [B^+(p), B^-(q)] = 2|p|\,\delta(p-q) + 2|p||q|\,\delta(p)\,\delta(q)\int_{-\kappa}^{\kappa}\frac{\mathrm{d}k}{|k|}$$

$$[A^+(p), A^-(0)] = [A^+(0), A^-(p)] = [B^+(p), B^-(0)] = [B^+(0), B^-(p)] = 2|p|\,\delta(p)$$

$$[A^+(0), A^-(0)] = [B^+(0), B^-(0)] = 0$$

$$[A^+(p), B^-(q)] = [B^+(p), A^-(q)] = 2\mathscr{P}\frac{1}{p}\delta(p-q)$$

$$[A^+(p), B^-(0)] = [B^+(p), A^-(0)] = 0 \qquad [A^+(0), B^-(0)] = [B^+(0), A^-(0)] = 0$$

$$[A^\pm(0), \phi^\mp(x)] = \pm 1/(2\pi)^{1/2} \qquad [B^\pm(0), \tilde{\phi}^\mp(x)] = \mp\frac{1}{(2\pi)^{1/2}}$$

$$[A^\pm(0), \tilde{\phi}^\mp(x)] = [A^\pm(0), \tilde{\phi}^\pm(x)] = [A^\pm(0), \phi^9(x)] = 0$$

$$[B^\pm(0), \phi^\mp(x)] = [B^\pm(0), \phi^\pm(x)] = [B^\pm(0), \tilde{\phi}^\pm(x)] = 0.$$

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